A Simple Analysis of Higher Order Liftings for Binary Problems

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Abstract

A very short and simple proof is presented showing that n-th lifting for the max-cut-polytope is exact. The proof is based on the observation that the max-cut-polytope is the projection of a higher-dimensional regular simplex and the fact that this simplex coincides with the n-th semidefinite lifting. An extension to include linear equality and inequality constraints concludes this note.

An extended version of this paper will appear in **Pacific Journal of Optimization**.

1 Introduction

Starting with the work of Gomory [5] an elegant general cutting plane procedure became available for solving integer linear programs. Unfortunately, even in two dimensions there is no universal bound on the number of cutting plane steps needed for solving an integer linear program. Also, as the cutting planes at each step remove the leading digits of selected inequality constraints, rounding errors systematically accumulate and thus necessitate an increasingly higher precision for the numerical computations. To overcome these shortcomings Sherali and Adams, [12], Lovász and Schrijver, [10], Balas, Ceria, and Cornuéjols[2], Lasserre, [6], and others proposed convex exact higher order liftings for hard combinatorial problems where the number of lifting steps is bounded in terms of the dimension of the problem and for which no systematic issues concerning numerical accuracy have been reported to date. From a theoretical point of view some of the shortcomings of Gomory-cuts could thus be eliminated. But the shortcomings are not eliminated in full as the dimensions of the liftings are large and refer to binary variables for which also the number of Gomory-cuts can be bounded in terms of the dimension of the problem.

Starting with a particularly simple example of a binary problem (without any constraints), namely the max-cut problem, a very simple analysis of higher order liftings is derived in Sections 2 and 3. In a second step linear constraints are added to this formulation.

1.1 Notation

Given $n \in \mathbb{N}$, let $N := \{1, \ldots, n\}$. For $I, J \subseteq N$ let $I \triangle J := (I \cup J) \backslash (I \cap J)$ be the symmetric difference of I and J. For $x \in \{\pm 1\}^n$ let $\vec{x} \in \mathbb{R}^{2^n}$ denote the augmented vector with components $\vec{x}_I := \prod_{i \in I} x_i$ for $I \subseteq N$.

By convention, when I is the empty set, $\vec{x}_\emptyset := 1$. Also note that for $I, J \subseteq N$, the product $\vec{x}_I \vec{x}_J$ is given by $\vec{x}_I \vec{x}_J = \vec{x}_{I \triangle J}$.

The space of real symmetric $k \times k$ matrices is denoted by $\mathcal{S}^k$ and the cone of real symmetric positive semidefinite $k \times k$ matrices is denoted by $\mathcal{S}^k_+$. The trace inner product of $X, Y \in \mathcal{S}^k$ inducing the Frobenius norm $\| \, . \, \|_F$ is denoted by $\langle X, Y \rangle$. The all-one-vector is denoted by e, its dimension being evident from the context. The max-cut-polytope is denoted by $\mathbf{MC} := \operatorname{conv}(\{xx^T \mid x \in \{\pm 1\}^n\})$, see e.g. [3, 4, 11].

2 A High-Dimensional Simplex

Consider the convex hull of all $\vec{x}\vec{x}^T$ where $\vec{x}$ is an augmented $\{\pm 1\}$-vector, i.e. consider the polytope

$$\mathbf{S} := \operatorname{conv}(\ \{ \ \vec{x}\vec{x}^T \mid x \in \{\pm 1\}^n \ \} \ ) \subset \mathcal{S}^{2^n}.$$

Proposition 2.1 *For $x, y \in \{\pm 1\}^n$ with $x \neq y$ it always follows $\vec{x}^T\vec{y} = 0$ and $\|\vec{x} - \vec{y}\|_2 = 2^{(n+1)/2}$. And for the associated vertices $\vec{x}\vec{x}^T$ and $\vec{y}\vec{y}^T$ of $\mathbf{S}$ it follows that $\langle \vec{x}\vec{x}^T, \vec{y}\vec{y}^T \rangle = 0$ and $\|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F = 2^{(2n+1)/2}$.*

Proof. Consider the case that y differs from x exactly in the components $1, \ldots, k$, i.e. $x_1 y_1 = \ldots = x_k y_k = -1$. As is well known and easy to verify[1] the number of subsets of $\{1, \ldots, k\}$ with an even number of elements is 2^{k-1} and the number of subsets of $\{1, \ldots, k\}$ with an odd number of elements also is 2^{k-1}. If an even number of elements from $\{1, \ldots, k\}$ is contained in $I \subseteq N$, then $\vec{x}_I = \vec{y}_I$, else $\vec{x}_I = -\vec{y}_I$. Similarly when some subset $\{i_1, \ldots, i_k\}$ of N is considered in place of $\{1, \ldots, k\}$. Hence exactly half the entries of $\vec{x}$ and $\vec{y}$ differ, so that $\vec{x}^T\vec{y} = 0$ and also $\langle \vec{x}\vec{x}^T, \vec{y}\vec{y}^T \rangle = \operatorname{trace}(\vec{x}\vec{x}^T\vec{y}\vec{y}^T) = 0$. This implies $\|\vec{x} - \vec{y}\|_2^2 = \|\vec{x}\|_2^2 + \|\vec{y}\|_2^2 = 2 \cdot 2^n = 2^{n+1}$ and likewise $\|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F^2 = 2^{2n+1}$. $\square$

Remark 2.1 *There are 2^n subsets $I \subseteq N$, and in the following, rows and columns of $X \in \mathcal{S}^{2^n}$ will be indexed by such subsets I. Let*

$$\mathcal{A} := \{X \in \mathcal{S}^{2^n} \ \mid \ X_{\emptyset,\emptyset} = 1, \ \ X_{I,J} = X_{K,L} \ \ \text{for any } I, J, K, L \subseteq N \text{ with } I \triangle J = K \triangle L\}. \quad (1)$$

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[1] Indeed for $n = 1$ the two subsets of N are $\emptyset$ and N. ✓

Now let $\hat{n} := n - 1 \geq 1$ and $\hat{N} := \{1, \ldots, \hat{n}\}$. The subsets of N are given by I and $I \cup \{n\}$ where $I \subseteq \hat{N}$. By induction hypothesis, $2^{\hat{n}-1}$ of the sets I and also $2^{\hat{n}-1}$ of the sets $I \cup \{n\}$ have even cardinality. Thus, the claim follows from $2^{\hat{n}-1} + 2^{\hat{n}-1} = 2^{n-1}$. ✓ $\square$

The linear equations in (1) relating $X_{I,J}$ and $X_{K,L}$ represent simple equalities that are satisfied for all vertices of $\mathbf{S}$ *such as*

$$\vec{x}_{\{i,j\}}\vec{x}_{\{i,k\}} = \vec{x}_{\{j\}}\vec{x}_{\{k\}} = \vec{x}_{\emptyset}\vec{x}_{\{j,k\}}.$$

As each vertex of $\mathbf{S}$ *satisfies the linear equations in (1) this implies that* $\mathbf{S} \subset \mathcal{A}$*. Since* $X_{\emptyset,\emptyset} = 1$*, it follows in particular that* $X_{I,I} = 1$ *for all* $I \subseteq N$ *so that there are only* $2^n - 1$ *"free" matrix entries* $X_{I,J}$ *of* X *in* $\mathcal{A}$ *depending on* $I \triangle J \neq \emptyset$*. Thus, the dimension of* $\mathcal{A}$ *is* $2^n - 1$*.*

Remark 2.2 *Within* $\mathcal{A}$ *the set* $\mathbf{S}$ *is a regular simplex with* 2^n *vertices. It is well known and easy to see*[2] *that* $\mathbf{S}$ *is centered about the identity matrix* $I \in \mathcal{S}^{2^n}$*. In particular,* $\mathbf{S}$ *is full-dimensional within* $\mathcal{A}$*. As shown in Proposition 2.1 the vertices are perpendicular to each other which seems to contradict the fact that the vertices of a full-dimensional regular simplex centered about the origin of some Euclidean space share an angle of more than 90 degrees to each other. However,* $\mathcal{A}$ *is embedded in a higher-dimensional space and does not contain the origin; instead all vertices* $\vec{x}\vec{x}^T$ *of* $\mathbf{S}$ *are centered about the identity matrix in* $\mathcal{A}$ *reducing the pairwise angle to 90 degrees.*

The first central observation used in this note is:

- The projection of $\mathbf{S}$ onto the rows and columns associated with $\vec{x}_{\{1\}}, \dots, \vec{x}_{\{n\}}$ is the max-cut-polytope $\mathbf{MC} = \text{conv}(\{xx^T \mid x \in \{\pm 1\}^n\})$ in $\mathcal{S}^n$.

The max-cut-polytope has 2^{n-1} vertices as x and $z := -x$ generate the same vertex, i.e. $xx^T = zz^T$, but $\vec{x}$ and $\vec{z}$ do not, i.e. $\vec{x}\vec{x}^T \neq \vec{z}\vec{z}^T$.

The second central observation used in this note (with a new short proof below) is that the semidefinite relaxation $\tilde{\mathbf{S}}$ of $\mathbf{S}$ coincides with the n-th lifting for the max-cut-polytope and also coincides with $\mathbf{S}$. This observation implies the known fact (established for example in [6, 10]) that the semidefinite liftings of sufficiently high order do represent the exact convex hull, and thus also the max-cut-polytope. The n-th lifting in [6, 10] can also be seen as a high dimensional linear extension of $\mathbf{MC}$ as introduced in Theorem 3 of [13]; the above representation of $\mathbf{S}$ via its 2^n facets is a very simple form of such linear extension.

2.1 Semidefinite Representation of the Simplex

Note that $\mathbf{S} \subseteq \tilde{\mathbf{S}}$, where the semidefinite relaxation $\tilde{\mathbf{S}}$ is given by

$$\tilde{\mathbf{S}} := \mathcal{S}_+^{2^n} \cap \mathcal{A}. \tag{2}$$

with $\mathcal{A}$ defined in (1). The set $\tilde{\mathbf{S}}$ is essentially the same as the n-th lifting for the max-cut-polytope defined in [1]. Same as $\mathbf{S}$, also $\tilde{\mathbf{S}}$ is contained in the $(2^n - 1)$-dimensional affine space $\mathcal{A}$. In fact, as shown next, both sets coincide.

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[2] Each vertex $X \in \mathbf{S}$ has an all-one-diagonal, and so does the average of all vertices. The off-diagonal elements of X are given by $X_{I,K} = \prod_{i \in I \triangle K} x_i$ so that among all $x \in \{\pm 1\}^n$, half of the entries $X_{I,K}$ are 1, and the average is zero.

Lemma 2.1 *The sets* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *coincide.*

Proof. Both, $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are full-dimensional, bounded, closed, convex subsets of the (2^n-1) dimensional affine subspace $\mathcal{A}$ of (1) containing the identity matrix I in $\mathcal{S}^{2^n}$ in their relative interior.

Next, it is shown that all relative boundary points $X \in \partial\mathbf{S}$ are also at the relative boundary of $\tilde{\mathbf{S}}$,

$$\partial\mathbf{S} \subset \partial\tilde{\mathbf{S}},$$

i.e. that all relative boundary points $X \in \partial\mathbf{S}$ have rank at most $2^n - 1$.

Indeed, let X be a boundary point of the simplex $\mathbf{S}$. Then, X is a convex combination of vertices of $\mathbf{S}$ with the exception of at least one vertex $\vec{y}\vec{y}^T$ of $\mathbf{S}$. Let $\vec{x}\vec{x}^T$ be some vertex different form $\vec{y}\vec{y}^T$. By Proposition 2.1,

$$2^{2n+1} = \|\vec{x}\vec{x}^T - \vec{y}\vec{y}^T\|_F^2 = \|\vec{x}\vec{x}^T\|_F^2 + \|\vec{y}\vec{y}^T\|_F^2 - 2(\vec{x}^T\vec{y})^2 = 2^{2n+1} - 2(\vec{x}^T\vec{y})^2$$

so that $(\vec{x}^T\vec{y})^2 = \vec{y}^T(\vec{x}\vec{x}^T)\vec{y} = 0$. As this is true for all other vertices $\vec{x}\vec{x}^T$ it follows that $\vec{y}^T X\vec{y} = 0$, i.e. X has rank at most $2^n - 1$.

Now, let $I \neq X \in \mathbf{S}$ be given. Since $\mathbf{S}$ is bounded the line starting at I and passing through X will cross the relative boundary of $\mathbf{S}$ at some point $\bar{X} \in \partial\mathbf{S} \subset \partial\tilde{\mathbf{S}}$. Since $\tilde{\mathbf{S}}$ is closed it follows $\bar{X} \in \tilde{\mathbf{S}}$ and by convexity also $X \in \tilde{\mathbf{S}}$. This shows $\mathbf{S} \subset \tilde{\mathbf{S}}$.

Conversely, let $\tilde{X} \in \tilde{\mathbf{S}}$ be given. If $\tilde{X} \notin \mathbf{S}$ the line segment from I to $\tilde{X}$ intersects the boundary of $\mathbf{S}$ at some point $\hat{X} \in \partial\mathbf{S} \subset \partial\tilde{\mathbf{S}}$. But $\hat{X}$ being in the open segment between I in the relative interior of $\tilde{\mathbf{S}}$ and $\tilde{X} \in \tilde{\mathbf{S}}$ cannot be at the boundary of $\tilde{\mathbf{S}}$. This contradiction shows that also $\tilde{\mathbf{S}} \subset \mathbf{S}$. ☐

Remark 2.3 *Some definitions of higher order liftings contain redundancies such as identical rows and columns. The set* $\tilde{\mathbf{S}}$ *is the* n*-th lifting after eliminating identical rows and columns. For* $1 \leq k < n$*, liftings of order* k *can be defined in a similar way by considering augmented vectors* $\vec{x}$ *with components* $\vec{x}_I$ *where* $I \subseteq N$ *has cardinality at most* k*. The corresponding semidefinite approximation of the max-cut-polytope is defined in an analogous way as the projection of the semidefinite relaxation for* $\vec{x}\vec{x}^T$ *onto rows and columns associated with* $\vec{x}_{\{1\}}, \ldots, \vec{x}_{\{n\}}$*. The previous lemma implies for any subset* $M \subset N$ *of cardinality at most* k *that the restriction of the* k*-th lifting to the matrix with entries* $X_{I,J}$ *for* $I, J \subseteq M$ *is exact, indicating that the accuracy of the lifting is improving when increasing* k*.*

To conclude the main part of this paper we note that the observation of Proposition 2.1 can also be found in [8], another derivation of Lemma 2.1 can be found in Lemma 8.15 in [9], and a simple proof of a result equivalent to Lemma 2.1 is also given in [2].

3 Including Linear Constraints

A key observation used in the approach of Lovász and Schrijver [10] concerns the inclusion of inequalities: Let a feasible set $\mathbb{P}$ be given by the convex hull of ± 1-vectors satisfying linear inequalities $(a^{(j)})^T x + \alpha_j \geq 0$ for $j \in M$ with some finite set M,

$$\mathbb{P} = \operatorname{conv}(\{x \mid x \in \{\pm 1\}^n, \ \ (a^{(j)})^T x + \alpha_j \geq 0 \text{ for } j \in M\}).$$

Then, since squared variables are identical to one, any finite product of these inequalities can be expressed as linear inequalities in terms of elements of the augmented vectors $\vec{x}$, the constant terms α_j being represented via $\vec{x}_\emptyset = 1$. Analogously, products of n inequalities of the form $(\pm x_i + 1) \geq 0$ for $1 \leq i \leq n$ can be represented as follows: For a vector $p \in \{0,1\}^n$ let the vector $\bar{\mathbf{p}} \in \mathbb{R}^{2^n}$ be defined by the identity

$$\bar{\mathbf{p}}^T \vec{x} \equiv \prod_{i=1}^{n} ((-1)^{p_i} x_i + \vec{x}_\emptyset) \quad \text{for any } x \in \{\pm 1\}^n \text{ and the associated } \vec{x} \in \{\pm 1\}^{2^n}.$$

($p_i = 0$ introduces a factor $(x_i + \vec{x}_\emptyset)$ and $p_i = 1$ introduces a factor $(-x_i + \vec{x}_\emptyset)$.)
In the next lemma it is observed that an exact relaxation of $\mathbb{P}$ is obtained when augmenting the components of the vectors $a^{(j)} \in \mathbb{R}^n$ for $j \in M$ to vectors

$$\bar{a}^{(j)} := (\alpha_j, (a^{(j)})^T, 0, \ldots, 0) \in \mathbb{R}^{2^n}$$

and when forming the semidefinite relaxation of order n with the same constraints as in (2) and with $2^n|M|$ additional constraints

$$\langle \bar{\mathbf{p}}(\bar{a}^{(j)})^T + \bar{a}^{(j)}(\bar{\mathbf{p}})^T, X \rangle \geq 0 \quad \text{for } j \in M \text{ and all } p \in \{0,1\}^n \text{ and their associated } \bar{\mathbf{p}}. \tag{3}$$

Lemma 3.1 *The first components $X_{\emptyset,\{1\}}, \ldots, X_{\emptyset,\{n\}}$ of the above semidefinite relaxation represent the exact convex hull $\mathbb{P}$ of all vectors $x \in \{\pm 1\}^n$ satisfying the linear inequalities $(a^{(j)})^T x + \alpha_j \geq 0$ for $j \in M$.*

Proof. For $M = \emptyset$ the all-one-diagonal of X and semidefiniteness of X imply the trivial fact that the vector $(X_{\emptyset,\{1\}}, \ldots, X_{\emptyset,\{n\}})^T$ is contained in the convex hull $[-1,1]^n$ of all $\{\pm 1\}$-vectors in $\mathbb{R}^n$. When $M \neq \emptyset$ assume that $\bar{x} \in \{\pm 1\}^n$ is a vector violating the constraint $(a^{(j)})^T \bar{x} + \alpha_j \geq 0$ for some $j \in M$. Selecting $p = (1 - \bar{x})/2$ in (3) and using

$$\bar{\mathbf{p}}^T \vec{\bar{x}} = \prod_{i=1}^{n} ((-1)^{p_i} \bar{x}_i + \vec{\bar{x}}_\emptyset) = 2^n > 0$$

it follows that (3) is violated by $X = \vec{\bar{x}} \vec{\bar{x}}^T$. On the other hand $\bar{\mathbf{p}}^T \vec{x} = 0$ for any other $x \in \{\pm 1\}^n$, since at least one of the factors $((-1)^{p_i} x_i + \vec{x}_\emptyset)$ is zero. Thus, (3) is satisfied with equality by any other vertex $\vec{x}\vec{x}^T$. Due to Lemma 2.1 any point in the semidefinite relaxation of order n is a convex combination of vertices $\vec{x}\vec{x}^T$ and by the above observation only vertices $\vec{x}\vec{x}^T$ satisfying all constraints occur in the convex combination. $\square$

Based on ideas from [10] a proof of Lemma 3.1 is also given in [7]. Note that the approach in [10] is slightly different: The $\{\pm 1\}$-formulation is replaced with the equivalent $\{0,1\}$-formulation, and the constraints $\pm x_i + 1 \geq 0$ are added to the constraints in M forming a set $\tilde{M}$ (with $|\tilde{M}| = 2n + |M|$). Then the $2^n M$ constraints in (3) are replaced with $|\tilde{M}|^n$ constraints

$$\prod_{\ell=1}^{n} ((a^{(j_\ell)})^T x + \alpha_{j_\ell}) \geq 0 \tag{4}$$

for all choices of $j_\ell \in \tilde{M}$. Again (4) can be expressed as linear constraints of the entries of X forming an extended lifting that includes all the constraints of the form (3). (Some of these constraints are redundant.)

When reducing the extended lifting of order n to an order less than n, the relaxation based on (4) is an improvement compared to (3), but Lemma 3.1 indicates that (3) is sufficient for the semidefinite lifting of order n.

Unfortunately, for all approaches, the dimensions of the liftings of order higher than one generally are too large to be computationally competitive, and therefore so far they are mostly of theoretical interest.

To conclude note that for linear equality constraints $(a^{(j)})^T x + \alpha_j = 0$ on the binary variable $x \in \{\pm 1\}^n$ the constraints (3) can be simplified (substantially) to just $|M|$ constraints

$$\langle \bar{a}^{(j)} (\bar{a}^{(j)})^T, X \rangle = 0 \quad \text{for } j \in M. \tag{5}$$

Indeed if (5) is satisfied, then by semidefiniteness of X the vectors $\bar{a}^{(j)}$ lie in the null space of X and then all constraints (3) are satisfied so that the argument in the proof of Lemma 3.1 remains valid.

4 Conclusion

This paper is intended as a reference providing a self-contained and simple proof of the known fact that the n-th semidefinite lifting of binary problems is exact. The unconstrained case of the max-cut-polytope is considered first and the results are then extended to reduced liftings with just rows corresponding to odd/even subsets of variables, a mixed reduced setting, and finally, the constrained case with both inequalities and equalities. A brief discussion of some related work concludes this paper.

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